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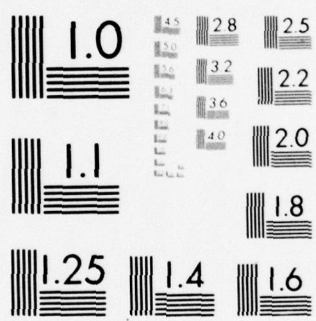

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In this paper we develop time-domain state space models for lossless layered media which are described by the wave equation and boundary conditions. Our models are for non-equal one-way travel times; hence, they are more general than existing models of layered media which are usually for layers of equal one-way travel times. Full state models, which involve 2K states for a K-layer media system, as well as half-state models, which involve only K states are developed and related. Certain transfer functions, which appear in the geophysics literature in connection with models of layered media with equal travel times, are generalized to the situation

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ABSTRACT

In this paper we develop time-domain state space models for lossless layered media which are described by the wave equation and boundary conditions. Our models are for non-equal one-way travel times; hence, they are more general than existing models of layered media which are usually for layers of equal one-way travel times. Full state models, which involve 2K states for a K-layer media system, as well as half-state models, which involve only K states are developed and related. Certain transfer functions, which appear in the geophysics literature in connection with models of layered media with equal travel times, are generalized to the situation of non-equal travel times. Our state space models represent a new class of equations, causal functional equations, some of whose properties and approaches to simulation are discussed.

I. INTRODUCTION

We are interested in lossless layered media which are described by the wave equation and boundary conditions. Specific applications of such media are: (1) horizontally stratified nonabsorptive earth with vertically traveling plane compressional waves, and (2) interconnection of lossless, not necessarily matched, transmission lines. Other applications can be found in acoustics, optical thin coatings, EM problems, etc. Because our interest is presently in the seismic area, our discussions will be in the context of such models.

A system of K layered media is depicted in Fig. 1. We adopt the convention of calling the layer below layer K the basement. The basement is assumed to act like an energy sink; i.e., no energy is returned from the basement into the K layers. Each layer is characterized by its one way travel time, τ_i, velocity, v_i, and normal incidence reflection coefficient r_i ($i = 1,2,\ldots,K$). Additionally interface-0 denotes the surface and is characterized by reflection coefficient r_0. In Fig. 1, $m(t)$ and $y(t)$ denote the input (e.g., seismic source signature from dynamite, airgun, etc.) to the layered media system which is applied at interface-0, and the output (i.e., ideal seismogram) of the system which is observed at the surface, respectively.*

We shall present state space models for the Fig. 1 system. These models, as we shall describe more fully below, are quite different from those which have appeared in the Geophysics literature (Refs. 1-4, for example). One big difference is that our models are for non-equal one-way travel times.

An important use of a model of a K-layer media system is to generate synthetic seismograms; i.e., to generate $y(t)$ for a given $m(t)$. This synthetic data can then be used for preliminary testing and evaluation of signal processing techniques (e.g., deconvolution). These models may also be useful for identifying important parameters, such as reflection coefficients and one-way travel times.**

As in Refs. 2 and 3, we shall find it convenient to draw ray diagrams with time displacement along the horizontal axis, so that the rays appear to be at non-normal incidence and so do not overlap one another. Figure 2 depicts primary and multiple reflections for a 2-layer media system, and illustrates the very complicated internal behavior of even a 2-layer system. Our state space models can not only be used to compute $y(t)$, but can also be used to compute the internal behavior of a layered media system. They are based on ray theory, which gives exact results for lossless, horizontally stratified media.

The starting point for our developments is the Fig. 3 ray diagram. Symbols u_k and d_k' denote the upgoing and downgoing waves in the kth layer, respectively; and, we adopt the convention that waves at the top of a layer occur at present time, t. That each layer is characterized by two signals traveling in opposite directions follows directly from the solution of a lossless wave equation. Geophysicists will recognize that Fig. 3 is also the starting point for the models which appear in Refs. 1 through 4, for example. We shall return to this point shortly.

As stated by Robinson (Ref. 3), "the solution of the wave equation at each interface leads to the definition of a reflection coefficient r_j associated with that interface. ...the reflection coefficient r_j, which must satisfy $|r_j| < 1$, has these properties. A downgoing wave of amplitude A in layer j, upon striking interface j, is both reflected and transmitted. The reflected portion is an upgoing wave of amplitude $r_j A$ in layer j, so r_j represents the reflection coefficient. The transmitted portion is a

*In a marine environment, layer 1 can be taken to be water; but, in that case $m(t)$ is applied and $y(t)$ is observed in the water layer. It is relatively straightforward to extend the results of this paper to that case.

**The reader interested in elements of the seismic prospecting method and the seismic reflection technique should see Ref. 5, Chs. 1 and 3, and Refs. 6, 7, and 8.

downgoing wave of amplitude $(1+r_j)A$ in layer j+1, so $1+r_j$ represents the transmission coefficient. An upgoing wave of amplitude B in layer j+1 is both reflected and transmitted when it strikes interface j. The reflected portion is a downgoing wave of amplitude $-r_j B$ in layer j+1, and the transmitted portion is an upgoing wave of amplitude $(1-r_j)B$. Hence $-r_j$ and $(1-r_j)$ represent, respectively, the reflection coefficient and transmission coefficient for the upgoing wave. These properties are summarized in Table 1 (2)."

Table 1. Reflected and Transmitted Portions

	Reflected Portion	Transmitted Portion
Downgoing wave A in layer j	Upgoing wave $r_j A$ in layer j	Downgoing wave $(1+r_j)A$ in layer j+1
Upgoing wave B in layer j+1	Downgoing wave $-r_j B$ in layer j+1	Upgoing wave $(1-r_j)B$ in layer j

Waveform $u_k(t+\tau_k)$ (Figure 3) is made up of two parts, namely the part due to the reflected portion of $d_k'(t-\tau_k)$ and the part due to the transmitted portion of $u_{k+1}(t)$. It satisfies the equation

$$u_k(t+\tau_k) = r_k d_k'(t-\tau_k) + (1-r_k)u_{k+1}(t). \quad (1)$$

In a similar manner, waveform $d_{k+1}'(t)$ satisfies the equation

$$d_{k+1}'(t) = (1+r_k)d_k'(t-\tau_k) - r_k u_{k+1}(t). \quad (2)$$

We refer to Eqs. (1) and (2) as the interface equations. These equations are the starting point for transfer function models, which are very popular in the Geophysics literature (Ref.2), and, they are also our starting point for the development of time-domain state space models (see Fig.4).

In the Geophysics literature, the assumption of equal one-way travel times, τ_i, is usually made. Layers of different travel times are built up by inserting layers whose reflection coefficients are zero. Transfer functions which relate the upgoing and downgoing signals from one layer to the next have been obtained. Because of the equal one-way travel time assumption, some very clever recursive equations have been developed (Refs.3,4, and 9, for example) which simplify the calculations of the transfer functions relating layers. These transfer functions are often presented in terms of z-transforms, which is again a consequence of the equal travel time assumption. From the transfer functions relating layers, it is possible to obtain the so-called reflection transfer function, $Y(s)/M(s)$ [or $Y(z)/M(z)$], which is used to generate synthetic seismogram data (Refs.3,4, and 9, for example). Additionally, Robinson (Ref.10) has presented a very simple technique for computing $Y(z)/M(z)$ in a layer-recursive manner. Inverse problems of reconstructing the reflection coefficients from knowledge of y(t) and m(t) have been extensively studied. Their solutions (Refs.3,4, and 11, for example) are again quite strongly dependent on the equal travel time assumption, and use z-transform relationships.

As is well known, there is a vast literature associated with systems which are described by time-domain state space models. Most recent results in estimation and identification theory, for example, require a state space model. In fact, our ultimate objective is to apply those theories to the layered media problem; but, to do so, of course requires state space models. One might argue that it should be possible to go directly from the transfer functions, already developed, to equivalent state space models. In most cases, this is not practical since closed-form expressions for the reflection transfer function are not available; that transfer function must be computed from a set of equations which are solved in a recirsive manner. Additionally, the transfer function results which appear in the Geophysics literature are limited by the equal travel-time assumption.

In this paper, we develop (see Fig.4) a variety of state models for the Fig.1 system. Full-state models, which are of dimension 2K (i.e., two states per layer), are described in Sections II and V.A. Half-state models, which are of dimension K (i.e., one state per layer), are described in Sections IV and V.B. A number of useful transfer function relationships are described in Section III. Those relationships are obtained directly from state equations, and, not only serve to connect our results with the transfer functions found in the Geophysics literature, but also represent generalizations of those results to the non-equal travel time case. Our state equations are continuous time equations with multiple time-delays, and, are referred to as causal functional equations. Some preliminary discussions on simulation of such equations is given in Section VI. Some new results which have been obtained using our state space models, as well as directions for future research, are given in Section VII.

II. A STATE EQUATION MODEL

A state equation model for our K layer media system is obtained directly from Eqs.(1) and (2), which are applicable for k=1,2,...,K-1, and comparable equations at the surface and Kth interface. At the surface (Fig.5a), we obtain

$$y(t) = r_0 m(t) + (1-r_0)u_1(t) \quad (3)$$

$$d_1'(t) = (1+r_0)\, m(t) - r_0 u_1(t); \quad (4)$$

and, at the Kth interface, we assume* that $u_{K+1}(t) = 0$, to obtain (Fig.5b)

$$u_K(t+\tau_K) = r_K d_K'(t-\tau_K) \quad (5)$$

$$d_{K+1}'(t) = (1+r_K)d_K'(t-\tau_K)\ . \quad (6)$$

Signal y(t) in Eq.(3) is the measurable system output. Signal $d_{K+1}'(t)$ is also a system output; but, since it cannot be measured, we shall ignore it in following analyses.

It is convenient to group Eqs.(1),(2),(4), and (5) in a layer ordering, as follows:

*This assumption is a boundary condition which is compatible with what is meant by the "basement."

$$d_1'(t)=-r_0u_1(t)+(1+r_0)m(t)$$
$$u_1(t+\tau_1)=r_1d_1'(t-\tau_1)+(1-r_1)u_2(t)$$
$$\left.\begin{aligned}d_j'(t)&=(1+r_{j-1})d_{j-1}'(t-\tau_{j-1})-r_{j-1}u_j(t)\\ u_j(t+\tau_j)&=r_jd_j'(t-\tau_j)+(1-r_j)u_{j+1}(t)\end{aligned}\right\}\ j=2,3,\ldots,K-1$$
$$d_K'(t)=(1+r_{K-1})d_{K-1}'(t-\tau_{K-1})-r_{K-1}u_K(t)$$
$$u_K(t+\tau_K)=r_Kd_K'(t-\tau_K)\ . \tag{7}$$

This system of 2K equations is not in a useful state equation format, yet, since signals in its left-hand side occur at t and delayed times, and signals on the right-hand side occur at t, $t-\tau_{j-1}$ and $t-\tau_j$. In order to put Eq.(7) into a useful state equation format, let

$$d_j(t) \overset{\Delta}{=} d_j'(t-\tau_j) \tag{8}$$

for all $j=1,2,\ldots,K$. Observe, from Fig.3, that the downgoing states $d_j(t)$ occur at the bottom of a layer. Equation (7) becomes

$$d_1(t+\tau_1)=-r_0u_1(t)+(1+r_0)m(t)$$
$$u_1(t+\tau_1)=r_1d_1(t)+(1-r_1)u_2(t)$$
$$\left.\begin{aligned}d_j(t+\tau_j)&=(1+r_{j-1})d_{j-1}(t)-r_{j-1}u_j(t)\\ u_j(t+\tau_j)&=r_jd_j(t)+(1-r_j)u_{j+1}(t)\end{aligned}\right\}\ j=2,3,\ldots,K-1$$
$$d_K(t+\tau_K)=(1+r_{K-1})d_{K-1}(t)-r_{K-1}u_K(t)$$
$$u_K(t+\tau_K)=r_Kd_K(t) \tag{9}$$

By means of transformation (8) each pair of equations in (7) now only involves two time points, $t+\tau_i$ and t. Equations (9) and (3) together represent the state equation model for the output y(t). This model is referred to as the <u>layer-ordered (L-O) full-state model</u> in the sequel.

Equations (9) and (3) can be expressed in more compact notation by introducing the following $2K\times 2K$ matrix operator*:

$$\mathcal{Z} \overset{\Delta}{=} \text{diag}(z_1,z_1,z_2,z_2,\ldots,z_K,z_K), \tag{10}$$

where z_i is a scalar operator used to denote a τ_i sec. time delay (i.e., $z_i f(t) = f(t-\tau_i)$). Let

$$\underline{\chi}(t)=\text{col}(u_1(t),d_1(t),u_2(t),d_2(t),\ldots,u_K(t),d_K(t)); \tag{11}$$

then, Eqs.(9) and (3) can be written, as

$$\mathcal{Z}^{-1}\underline{\chi}(t) = A\underline{\chi}(t) + \underline{b}m(t) \tag{12}$$
$$y(t) = \underline{c}'\underline{\chi}(t) + r_0m(t) \tag{13}$$

where the explicit structures of A, $\underline{b}$, and $\underline{c}$ can be deduced directly from the former equations. Because we do not need this information at this point, and because other versions of Eqs.(12) and (13), which we discuss in Section V-A, have more easily remembered A, $\underline{b}$, and $\underline{c}$ matrices, we do not give explicit A, $\underline{b}$, and $\underline{c}$ structures here for Eqs.(12) and (13). From Eqs.(12) and (13), we see that

$$\underline{\chi}(t)=(\mathcal{Z}^{-1}-A)^{-1}\underline{b}m(t) = (I-\mathcal{Z}A)^{-1}\mathcal{Z}\underline{b}m(t) \tag{14}$$

and

$$y(t)=[\underline{c}'(\mathcal{Z}^{-1}-A)^{-1}\underline{b}+r_0]m(t)=[\underline{c}'(I-\mathcal{Z}A)^{-1}\mathcal{Z}\underline{b}+r_0]m(t). \tag{15}$$

These equations provide us (conceptually, at least) with the solution of the state equation and with the output as a function of the input. The <u>transfer function</u> of the K layer media system is obtained directly from Eq.(15), as

$$\frac{Y(s)}{M(s)} = \underline{c}'(\hat{\mathcal{Z}}^{-1}-A)^{-1}\underline{b}+r_0 = \underline{c}'(I-\hat{\mathcal{Z}}A)^{-1}\hat{\mathcal{Z}}\underline{b}+r_0 \tag{16}$$

where $\hat{\mathcal{Z}}$ is obtained from $\mathcal{Z}$ by setting

$$z_i = e^{-s\tau_i}. \tag{17**}$$

At this point, some comments on the nature of state equation (12) are in order. In the special case when $\tau_1=\tau_2=\cdots=\tau_K \overset{\Delta}{=} \tau$, $\mathcal{Z} = zI$, where z denotes the τ sec time delay and I is the $2K\times 2K$ identity matrix; and, Eq.(12) can be written, as

$$\underline{\chi}(t+\tau) = A\underline{\chi}(t) + \underline{b}m(t)\ . \tag{18}$$

This equation can be reduced to a vector finite-difference equation by choosing $t=k\tau$, and, when m(t) only has values at $t=k\tau$. Then, all of the usual techniques associated with such equations can be used to analyze our K-layer media system. We do not choose to follow this uniform travel time/sampled data path, because these assumptions seem too restrictive.

State Equation (12) is a dynamical equation with multiple time delays. It is not a differential equation, nor is it a finite-difference equation. We shall refer to it as a <u>causal functional equation</u>. It is linear and time-invariant, and, as is the case with delay-time systems, requires initial value information over initial intervals of time. Equation (16) suggests a straightforward way to compute y(t) for an arbitrary m(t). First compute the system's impulse response, H(s), where, obviously

$$H(s) = \underline{c}'(I-\hat{\mathcal{Z}}A)^{-1}\hat{\mathcal{Z}}\underline{b}+r_0 \tag{19}$$

then, convolve h(t) with m(t) to obtain y(t). It is interesting to note that <u>h(t) is a sequence of impulse functions</u>, since the right-hand side of Eq. (19) is an infinite series each of whose terms looks like $\alpha e^{-s\beta}$, and, $\mathcal{L}^{-1}\{\alpha e^{-s\beta}\} = \alpha\delta(t-\beta)$.

Since our K layer media system is one with time delays, its state space is infinite-dimensional; but interestingly enough, only a finite number of states (i.e., 2K) are needed to describe the trajectories

*This idea was first suggested to us by Mr. Michael Steinberger, a graduate student in the Electrical Engineering Department, at the University of Southern California.

**The right-hand side of Eq.(17) is the Laplace transform of a delay effect. Associating z_i with a delay is common in the Geophysics' and time-series' literatures. In the control's literature, on the other hand, z_i is usually associated with an advance in which case $z_i=e^{s\tau_i}$. Clearly, our z_i is simply the inverse of z_i in the control's literature.

in that state space.

The important system theoretic concepts of observability, controllability, and identifiability can be defined in a variety of ways, as for differential-delay equations, and will be discussed elsewhere.

The L-O full-state model was first presented by Nahi and Mendel in Ref.12.

III. SOME TRANSFER FUNCTIONS

While Eq.(16) is the transfer function for our K-layer media system, it is not at all a useful form for computing that quantity. In this section we present two alternatives to Eq.(16), both of which are recursive in nature and are of interest in their own right.

A. Layer Transfer Functions

Let $\underline{X}_k(s)$ denote a 2×1 Laplace transformed vector, defined for the kth layer, as

$$\underline{X}_k(s) = \mathrm{col}[U_k(s), D_k(s)] \quad . \tag{20}$$

In Appendix A, we show that

$$\underline{X}_k(s) = \prod_{i=k}^{1} [W_i^{-1}(s) G_i(s)] M(s) \tag{21}$$

and

$$\frac{Y(s)}{M(s)} = (1-r_0)(1,0) W_1^{-1}(s) G_1(s) + r_0 \tag{22}$$

where $W_i(s)$ is a 2×2 matrix which is solved in a backwards recursive manner from the following algorithm:

$$\left.\begin{aligned} W_K(s) &= F_K(s) \\ W_i(s) &= F_i(s) - H_i(s) W_{i+1}^{-1} G_{i+1}(s) \\ i &= K-1, K-2, \ldots, 1 \end{aligned}\right\} \tag{23}$$

where

$$F_i(s) = \begin{pmatrix} 1 & -r_i z_i \\ r_{i-1} z_i & 1 \end{pmatrix} \tag{24}$$

$$H_i(s) = \begin{pmatrix} (1-r_i) z_i & 0 \\ 0 & 0 \end{pmatrix} \tag{25}$$

$$G_i(s) = \begin{pmatrix} 0 & 0 \\ 0 & (1+r_{i-1}) z_i \end{pmatrix} \tag{26}$$

and

$$G_1(s) = \begin{pmatrix} 0 \\ (1+r_0) z_1 \end{pmatrix} \tag{27}$$

In practice, it is not necessary to compute the four elements of $W_{i+1}^{-1}(s)$, due to the sparse nature of $H_i(s)$ and $G_{i+1}(s)$. Only the 1-2 element of $H_i W_{i+1}^{-1} G_{i+1}$ is non-zero, and that element depends only on the 1-2 element of $W_{i+1}^{-1}(s)$.

During the development of Eq.(21) we obtain $\underline{X}_j(s)$ as a function of $\underline{X}_{j-1}(s)$:

$$\left.\begin{aligned} \underline{X}_j(s) &= W_j^{-1}(s) G_j(s) \underline{X}_{j-1}(s) \ , \ j=2,3,\ldots,K \\ \underline{X}_1(s) &= W_1^{-1}(s) G_1(s) M(s) \end{aligned}\right\} \tag{28}$$

Matrix $W_j^{-1}(s) G_j(s)$ is the <u>layer transfer function matrix</u> which relates layers j-1 and j. Equation (28) is similar to the recursive algorithms which appear in the Geophysics literature (Refs.3,4 and 9, for example), except that it is a generalization of those algorithms to non-equal travel times.

We conclude this paragraph with an example which illustrates the calculation of $Y(s)/M(s)$ for K=2. In that case,

$$W_2 = F_2 = \begin{pmatrix} 1 & -r_2 z_2 \\ r_1 z_2 & 1 \end{pmatrix} \tag{29a}$$

$$W_2^{-1}(1,2) = \frac{r_2 z_2}{1+r_1 r_2 z_2^2} \tag{29b}$$

$$W_1 = F_1 - H_1 W_2^{-1} G_2 \begin{pmatrix} 1 & \dfrac{-(r_1+r_2 z_2^2) z_1}{1+r_1 r_2 z_2^2} \\ r_0 z_1 & 1 \end{pmatrix} \tag{29c}$$

$$W_1^{-1}(1,2) = \frac{(r_1+r_2 z_2^2) z_1}{1+r_1 r_2 z_2^2 + r_0 r_1 z_1^2 + r_0 r_2 z_1^2 z_2^2} \tag{29d}$$

and

$$\begin{aligned} \frac{Y(s)}{M(s)} &= r_0 + (1-r_0^2) z_1 W_1^{-1}(1,2) \\ &= \frac{r_0 + r_0 r_1 r_2 z_2^2 + r_1 z_1^2 + r_2 z_1^2 z_2^2}{1 + r_1 r_2 z_2^2 + r_0 r_1 z_1^2 + r_0 r_2 z_1^2 z_2^2} \ . \end{aligned} \tag{30}$$

For the special case where $\tau_1 = \tau_2$, so that $z_1 = z_2 \overset{\Delta}{=} z$, Eq.(30) simplifies to

$$\frac{Y(s)}{M(s)} = \frac{r_0 + (r_1 + r_0 r_1 r_2) z^2 + r_2 z^4}{1 + (r_0 r_1 + r_1 r_2) z^2 + r_0 r_2 z^4} \tag{31}$$

which is precisely the same result derived by Robinson in Ref.3.

B. Recursive Reflection Transfer Function Relationship

For a K-layer media system, states $u_j(t)$ and $d_j(t)$ have been defined at the top and at the bottom, respectively, of the jth layer (see Fig.3). Let $R_j(s)$ denote the transfer function between u_j and d_j <u>at the jth interface</u>; i.e.,

$$R_j(s) = \frac{\mathcal{L}\{u_j(t+\tau_j)\}}{\mathcal{L}\{d_j(t)\}} = e^{s\tau_j} \frac{U_j(s)}{D_j(s)} \ . \tag{32}$$

When j=0, we obtain the reflection transfer function between output $y(t)$ and source $m(t)$; i.e.,

$$R_0(s) = e^{s\tau_0}\frac{U_0(s)}{D_0(s)} \triangleq \frac{Y(s)}{M(s)}\,, \tag{33}$$

since $\tau_0=0$.

We shall now develop a simple recursive relationship between $R_j(s)$ and $R_{j+1}(s)$. Consider our earlier state equations for u_j and d_{j+1}:

$$u_j(t+\tau_j) = r_j d_j(t)+(1-r_j)u_{j+1}(t) \tag{34}$$

$$d_{j+1}(t+\tau_{j+1}) = (1+r_j)d_j(t)-r_j u_{j+1}(t)\,. \tag{35}$$

From the Laplace transform of Eq.(34), we find

$$R_j(s) = r_j+(1-r_j)U_{j+1}(s)/D_j(s) \tag{36}$$

Laplace transform Eq.(35) and solve for $D_j(s)$, to show that

$$D_j(s) = [r_j U_{j+1}(s)+e^{s\tau_{j+1}}D_{j+1}(s)]/(1+r_j) \tag{37}$$

Substitute Eq.(37) into Eq.(36), and re-arrange some terms in the resulting expression to see that

$$R_j(s) = \frac{r_j+z_{j+1}^2R_{j+1}(s)}{1+r_jz_{j+1}^2R_{j+1}(s)} \quad (j=K-1,K-2,\ldots,1,0) \tag{38}$$

which is the desired result.

Equation (38) can be used to compute the output of a K-layer media system in a recursive manner, beginning with a one layer system (i.e., one layer on top of a basement layer) for which we set $j=K-1$. We then *iterate* Eq.(38) *backwards*, *setting* $j=K-2$, $K-3,\ldots,1,0$. In order to compute $R_{K-1}(s)$ we need $R_K(s)$; but, $R_K(s)$ can be obtained directly from the very last state equation in (9), $u_K(t+\tau_K)=r_K d_K(t)$, as

$$R_K(s) = r_K\,. \tag{39}$$

In the special, but widely studied case of <u>equal travel times</u>, Eqs.(38) and (39) simplify to

$$\left.\begin{aligned} R_j(s) &= \frac{r_j+z^2R_{j+1}(s)}{1+r_jz^2R_{j+1}(s)}\,, \quad j=K-1,K-2,\ldots,1,0 \\ R_K(s) &= r_K \end{aligned}\right\} \tag{40}$$

where $z=e^{-s\tau}$. Equation (40) (or its discrete-time counterpart, in which Laplace transfer functions are replaced by z-transform transfer functions) is a well-known result which can be derived by widely different methods (Refs.10 and 13, for example). Additionally, these recursive relationships occur (Ref. 14) in electric kernal functions, magnetotelluric input impedence functions, and electromagnetic modified kernal functions.

That Eq.(40) generalizes to Eq.(38) for non-equal travel times is believed to be a new result.

To illustrate the use of Eqs.(38) and (39), we recompute $Y(s)/M(s)=R_0(s)$ for $K=2$. In that case, $R_2(s)=r_2$, and

$$R_1(s) = \frac{r_1+z_2^2r_2}{1+r_1r_2z_2^2}\,, \tag{41a}$$

and

$$\begin{aligned} R_0(s) &= \frac{r_0+z_1^2R_1(s)}{1+r_0z_1^2R_1(s)} \\ &= \frac{r_0+r_0r_1r_2z_2^2+r_1z_1^2+r_2z_1^2z_2^2}{1+r_1r_2z_2^2+r_0r_1z_1^2+r_0r_2z_1^2z_2^2}\,. \end{aligned} \tag{41b}$$

It is much easier to compute the reflection transfer function by the recursive reflection transfer function relationship of this paragraph than by the layer transfer functions of the preceding paragraph; however, detailed information about the upgoing and downgoing states cannot be recovered from the relationships in this paragraph, whereas they can be recovered from Eq.(21) in the preceding paragraph.

IV. <u>A RECURSIVE HALF-STATE MODEL</u>

Observe, from Eq.(38), that operator z_{j+1} appears only as z_{j+1}^2 in $R_j(s)$. This suggests that a state space model, which requires only K states, can be developed from that equation. We refer to this model as a <u>recursive half-state model</u>, since it is obtained from the recursive reflection transfer function relationship.

<u>Theorem 1</u>. For the K-layer media system depicted in Fig.1, let

$$\underline{x}(t) = \mathrm{col}(x_1(t),x_2(t),\ldots,x_K(t))\,, \tag{42}$$

$$Z = \mathrm{diag}(z_1,z_2,\ldots,z_K)\,, \tag{43}$$

and

$$T = (\underline{e}_K,\underline{e}_{K-1},\ldots,\underline{e}_1) \tag{44}$$

where $\underline{e}_j$ is the j-th unit vector, and is $K\times 1$. Then

$$TZ^{-2}T\underline{x}(t) = A\underline{x}(t)+\underline{b}m(t) \tag{45}$$

$$y(t) = \underline{c}'\underline{x}(t)+r_0m(t) \tag{46}$$

where

$$\underline{b} = \mathrm{col}(r_K,r_{K-1},\ldots,r_1) \tag{47}$$

$$\underline{c} = \mathrm{col}(0,0,\ldots,0,(1-r_0^2)) \tag{48}$$

and

$$A = \begin{pmatrix} -r_Kr_{K-1} & -r_Kr_{K-2} & -r_Kr_{K-3} & \cdots & -r_Kr_1 & -r_Kr_0 \\ (1-r_{K-1}^2) & -r_{K-1}r_{K-2} & -r_{K-1}r_{K-3} & \cdots & -r_{K-1}r_1 & -r_{K-1}r_0 \\ 0 & (1-r_{K-2}^2) & -r_{K-2}r_{K-3} & \cdots & -r_{K-2}r_1 & -r_{K-2}r_0 \\ 0 & 0 & (1-r_{K-3}^2) & \cdots & -r_{K-3}r_1 & -r_{K-3}r_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (1-r_1^2) & -r_1r_0 \end{pmatrix} \tag{49}$$

The proof of this theorem, which contains a recur-

sive version of Eqs.(45) and (46), is given in Appendix B. ∎

The left-hand side of Eq.(45) is a reverse double-delay operator; i.e.,

$$TZ^{-2}T\underline{x}(t)=\mathrm{col}(x_1(t+2\tau_K),x_2(t+2\tau_{K-1}),\ldots,x_K(t+2\tau_1)). \qquad (50)$$

Additionally, matrix T is a special permutation matrix, with the property $T^{-1}=T$; hence, Eq.(45) can also be written, as

$$\underline{x}(t) = TZ^2TA\underline{x}(t)+TZ^2T\underline{b}m(t) \ . \qquad (51)$$

In the special case of equal travel times, Z=zI, and Eq.(51) reduces to

$$\underline{x}(t) = A\underline{x}(t-2\tau)+\underline{b}m(t-2\tau) \ . \qquad (52)$$

Finally, whereas the recursive half-state model requires only K states, we at present can give no physical meaning to those states. In Section V.B, however, we relate the $\underline{x}$-states to the upgoing states $u_1(t),u_2(t),\ldots,u_K(t)$.

V. EQUIVALENT MODELS

In this section we present alternative, but equivalent, full-state and half-state models. The alternative full-state models are useful because they have much more revealing structures than the L-O full-state model, and one of these models — the skew model — leads quite naturally to a skew half-state model. That half-state model, whose states are some of the upgoing and some of the downgoing states from the L-O full-state model, can be related to the *Section IV recursive half-state model. In this way,* we give physical meanings to the states in the recursive half-state model.

A. Full-State Models

1. D-U Model

The D-U full-state model is obtained from the L-O full-state model by reordering the latter's equations in such a manner that all downgoing states are grouped together and all upgoing states are grouped together. Let

$$\underline{d}(t) = \mathrm{col}(d_1(t),d_2(t),\ldots,d_K(t)) \qquad (53)$$

and

$$\underline{u}(t) = \mathrm{col}(u_1(t),u_2(t),\ldots,u_K(t)) \ . \qquad (54)$$

Equations (9) and (3) can be written, in partitioned form as:

$$Z^{-1}\underline{d}(t) = A_1\underline{d}(t)+A_2\underline{u}(t)+\underline{g}m(t) \qquad (55a)$$

$$Z^{-1}\underline{u}(t) = A_3\underline{d}(t)+A_4\underline{u}(t) \qquad (55b)$$

and

$$y(t) = \underline{h}'\underline{u}(t)+r_0m(t) \qquad (56)$$

where

$$A_1=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ (1+r_1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & (1+r_2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & (1+r_3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & (1+r_{K-1}) & 0 \end{pmatrix} \qquad (57a)$$

$$A_2 = -\mathrm{diag}(r_0,r_1,\ldots,r_{K-1}) \qquad (57b)$$

$$A_3 = \mathrm{diag}(r_1,r_2,\ldots,r_K) \qquad (57c)$$

$$A_4=\begin{pmatrix} 0 & (1-r_1) & 0 & 0 & \cdots & 0 \\ 0 & 0 & (1-r_2) & 0 & \cdots & 0 \\ 0 & 0 & 0 & (1-r_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & (1-r_{K-1}) \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad (57d)$$

and

$$\underline{g} = \mathrm{col}(1+r_0,0,0,\ldots,0) \qquad (58)$$

$$\underline{h} = \mathrm{col}(1-r_0,0,0,\ldots,0) \ . \qquad (59)$$

Matrices A_1,A_2,A_3, and A_4 are $K\times K$, and matrix operator Z is defined in Eq.(43).

Equations (55a), (55b), and (56) comprise the D-U model. Matrices A_1,A_2,A_3,A_4 are easily remembered for this model. The D-U model has found application in Mendel's Bremmer series decomposition of output y(t) (Ref.15). That decomposition is discussed in Section VII.

2. Skew Model

The skew full-state model is also obtained from the L-O full-state model by a reordering or equations. Let $\underline{x}_{s1}(t)$ and $\underline{x}_{s2}(t)$ both be $K\times 1$ vectors, where

$$\underline{x}_{s1}(t) = \mathrm{col}(u_1(t),d_2(t),u_3(t),\ldots) \qquad (60)$$

and

$$\underline{x}_{s2}(t) = \mathrm{col}(d_1(t),u_2(t),d_3(t),\ldots). \qquad (61)$$

When K is even the last elements of $\underline{x}_{s1}(t)$ and $\underline{x}_{s2}(t)$ are $d_K(t)$ and $u_K(t)$, respectively; whereas, when K is odd, those elements are $u_K(t)$ and $d_K(t)$. Equations (9) and (3) can be expressed in terms of $\underline{x}_{s1}(t)$ and $\underline{x}_{s2}(t)$, as

$$Z^{-1}\underline{x}_{s1}(t) = G\underline{x}_{s2}(t) \qquad (62a)$$

$$Z^{-1}\underline{x}_{s2}(t) = H\underline{x}_{s1}(t)+\underline{g}m(t) \qquad (62b)$$

and

$$y(t) = \underline{h}'\underline{x}_{s1}(t)+r_0m(t) \qquad (63)$$

where

$$\begin{array}{ll} K = & \begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \end{array} \\ G = & \begin{pmatrix} r_1 & 1-r_1 & 0 & 0 & \cdots \\ 1+r_1 & -r_1 & 0 & 0 & \cdots \\ 0 & 0 & r_3 & 1-r_3 & \cdots \\ 0 & 0 & 1+r_3 & -r_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \qquad (64)$$

$$
\begin{array}{c c}
K = & \begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \end{array} \\
H = & \left(\begin{array}{ccccc}
-r_0 & 0 & 0 & 0 & \cdots \\
0 & r_2 & 1+r_2 & 0 & \cdots \\
0 & 1-r_2 & -r_2 & 0 & \cdots \\
0 & 0 & 0 & r_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
\end{array}
\qquad (65)
$$

and $\underline{g}$ and $\underline{h}$ are defined in Eqs.(58) and (59), resepctively. Matrices G and H are both $K \times K$, and are given above as functions of K.

Equations (62a), (62b), and (63) comprise the skew model. *It is called a skew model because only cross coupling terms appear on the right-hand side of the state equations;* i.e., the rather unusual ordering of our 2K states in $\underline{x}_{s1}(t)$ *and* $\underline{x}_{s2}(t)$ *leads to a partitioned state equation with blocks of zeros along the main diagonal.*

B. Half-State Models

1. Skew Model

Observe, from Eqs.(63) and (62) that y(t) depends only on $\underline{x}_{s1}(t)$, and that $\underline{x}_{s2}(t)$ can easily be eliminated from the state equations to give the following half-state state equation for $\underline{x}_{s1}(t)$:

$$\underline{x}_{s1}(t) = ZGZH\underline{x}_{s1}(t) + ZGZ\underline{g}m(t) \quad . \qquad (66)$$

Equations (66) and (63) constitute our skew half-state model. Half-state vector $\underline{x}_{s2}(t)$ can be computed *from Eq.(62b) once* $\underline{x}_{s1}(t)$ *has been computed.*

It is interesting to compare Eq.(66) and (51). Observe that the right-hand sides of these state equations involve matrix operators ZGZ and TZ^2T, respectively. If a physical element existed for implementing a delay operator matrix, then the skew half-state model would need two such elements, whereas the recursive half-state model would only need one such element [for Z^2]. This suggests that the recursive half-state model is minimal in terms of "hardware" requirements.

In the special case of equal travel times, where Z= zi, Eq.(66) reduces to

$$\underline{x}_{s1}(t) = GH\underline{x}_{s1}(t-2\tau) + G\underline{g}m(t-2\tau) \qquad (67)$$

which is comparable to Eq.(52). We see, therefore, that only in the case of equal travel times do both the skew and recursive half-state models have the same "hardware" requirements, in that they both require a double delay element, 2τ.

2. Upgoing States

Observe from Eq.(56) that y(t) depends only on $\underline{u}(t)$. While it is possible to eliminate $\underline{d}(t)$ from Eqs.(55a) and (55b) to obtain a half-state model for $\underline{u}(t)$, that model is not terribly useful, due to its apparent complexity.*

It is useful to obtain a direct relationship between $\underline{u}(t)$ and $\underline{x}_{s1}(t)$. In Appendix C, we show that

$$\underline{u}(t) = (L_a'L_a + L_b'L_b ZH)\underline{x}_{s1}(t) \qquad (68)$$

where L_a *and* L_b *are permutation matrices which* depend on whether K is even or odd. These matrices are also defined in Appendix C.

From Eqs.(68) and (66), one can develop a different half-state equation for $\underline{u}(t)$; but, that equation also does not appear to be terribly useful, so we shall omit it here. Clearly, we can compute u(t) by first computing $\underline{x}_{s1}(t)$ and then using Eq.(68).

It is also possible to relate $\underline{u}(t)$ *to* $\underline{x}(t)$. Washburn (Ref.16) has proven the following:

Theorem 2. Let

$$\gamma_j = \prod_{i=0}^{j-1} (1+r_i) \qquad (69)$$

and

$$\overline{T}(j) = \sum_{i=0}^{j-1} \tau_i \ , \quad \tau_0 \triangleq 0 \ . \qquad (70)$$

Then,

$$x_{K+1-j}(t) = \frac{1}{\gamma_j} u_j(t+\overline{T}(j)) \ , \quad j=1,2,\ldots,K \ . \qquad (71)$$

Proof: The proof of this theroem is inductive in nature and algebraically lengthy. The general idea is to show that $\underline{\zeta}(t) \triangleq \underline{\text{col}}(\zeta_K(t), \zeta_{K-1}(t), \ldots, \zeta_1(t))$, where $\zeta_{K-j}(t) = u_{K-j}(t+\overline{T}(K-j))/\gamma_{K-j}$, satisfies the recursive half-state equation (45); i.e., to show that for $\underline{\zeta}(t)$ as just defined, $TZ^{-2}T\underline{\zeta}(t) = A\underline{\zeta}(t) + \underline{b}m(t)$. By uniqueness, then, $\underline{\zeta}(t) = \underline{x}(t)$. Details can be found in Washburn (Ref.16). ■

It is useful to express Eq.(71) in operator notation. Let

$$\Gamma = \text{diag}(\gamma_K, \gamma_{K-1}, \ldots, \gamma_1) \qquad (72)$$

and

$$N = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \prod_{i=1}^{K-1} z_i \\
0 & 0 & 0 & \cdots & \prod_{i=1}^{K-2} z_i & 0 \\
\vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\
0 & 0 & z_1 z_2 & \cdots & 0 & 0 \\
0 & z_1 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \qquad (73)$$

then, Eq.(71) can be written as

$$\underline{x}(t) = \Gamma^{-1}N\underline{u}(t) \quad . \qquad (74)$$

Theorem 2 is a very useful result; for, not only does it give us another way to compute $\underline{u}(t)$ [i.e., $\underline{u}(t) = N^{-1}\Gamma\underline{x}(t)$], but it also provides us with meaning for the mathematically defined half-state vector, $\underline{x}(t)$, in terms of the physical half-state, $\underline{u}(t)$.

There is an interesting pictorial description of the relationships between $x_{K+1-j}(t)$ and $u_j(t)$ (j= 1,2,...,K). It is obtained by writing Eq.(71) as

$$u_j(t) = \gamma_j x_{K+1-j}(t-\overline{T}(j)) \ , \quad j=1,2,\ldots,K \qquad (75)$$

and is shown in Fig.6, for K=4. On that figure, we

*$\underline{u}(t) = Z[A_4 + A_3(I-ZA_1)^{-1}ZA_2]\underline{u}(t) + ZA_3(I-ZA_1)^{-1}Z\underline{b}m(t)$.

use the fact that $\gamma_j = \prod_{i=0}^{j-1} t_i$, where t_i is the ith transmission coefficient (see Table 1). States $x_1(t)$ through $x_4(t)$ are located at the surface of the 4-layer media system, as well they should be since they are associated with an input-output model. The dashed lines denote a fictitious ray path linking $x_{K+1-j}(t)$ and $u_j(t)$; and, of course, we know where the $u_j(t)$ states are located, since they are physical states. The relationship between u_j and x_{K+1-j} is one from ray theory. If, for example, $x_2(t)$ is applied at the surface then, following the direct transmission path between $x_2(t)$ and $u_3(t)$, it follows, from ray theory, that $u_3(t) = t_0 t_1 t_2 x_2(t-\tau_1-\tau_2)$.

In retrospect, Fig.6 provides a heuristic proof of Theorem 2.

3. Downgoing States

Suppose we desire information about downgoing states $\underline{d}(t)$. It is straightforward to show, that

$$\underline{d}(t) = (L_a' L_a + L_b' L_b ZG)\underline{x}_{s2}(t) \quad . \tag{76}$$

A procedure for computing $\underline{d}(t)$ is: (1) compute $\underline{x}_{s1}(t)$ using Eq.(66); (2) compute $\underline{x}_{s2}(t)$ from Eq.(62b); and (3) compute $\underline{d}(t)$ from Eq.(76).

VI. COMPUTATION

Our full-state and half-state models are causal functional equations, a class of equations about which we have not been able to find very much either in the system's or mathematics's literature. In this section we give some brief preliminary thoughts on simulation of causal functional equations.

A. Full-State Model

Our discussions in this paragraph are directed at simulation of the L-O full-state model. They are applicable, with modifications, to the D-U and skew full-state models.

A brute force approach to digital simulation is to: discretize time (our independent variable), map each τ_j in some unique manner into a quantum zone along the discretized time-axis, and then solve the resulting discrete-time system by means of finite-difference equation techniques. Appreciable errors may be introduced in the τ_j-mapping step, unless a very fine quantization is used. To-date, we have not tried this approach; but, we have not written it off either, since it is quite simple to implement.

A second approach, which we have begun to study in some detail, is one in which the system's impulse response is computed by* a branching process/table look-up procedure. Output, y(t), is then computed via convolution between m(t) and h(t) [see Section II]. The branching process/table look-up procedure is based on the observation that the basic operations required to compute the impulse response are shifting and adding of two non-uniform sequences of impulses. An alternate way of viewing Eqs.(3) and (9) is to consider what happens to $d_j(t)$ and $u_j(t)$. A careful study of these equations leads to the following transformation rules:

$$d_j(t) \begin{cases} r_0 d_0(t) \rightarrow y(t) & j=0 \\ r_j d_j(t) \rightarrow u_j(t+\tau_j) & j=1,2,\ldots,K \\ (1+r_j)d_j(t) \rightarrow d_{j+1}(t+\tau_{j+1}) & j=0,1,\ldots,K-1 \end{cases} \tag{77a}$$

$$u_j(t) \begin{cases} (1-r_0)u_1(t) \rightarrow y(t) & j=1 \\ (1-r_{j-1})u_j(t) \rightarrow u_{j-1}(t+\tau_{j-1}) & j=2,3,\ldots,K \\ -r_{j-1}u_j(t) \rightarrow d_j(t+\tau_j) & j=1,2,\ldots,K \end{cases} \tag{77b}$$

We search along the time axis for a time at which an event (i.e., an impulse) has occurred. At that time point, we map all d_j and u_j states according to Eqs.(77a) and (77b). Since the right-hand sides of these equations involve two time shifts, a single event branches into two events. We proceed along the time axis looking up values of d_j and u_j at event points, until we have covered the domain of interest. To eliminate costly computation, we set a lower bound on state amplitudes, below which we assume it to be zero. Additionally, if two events occur within a prespecified tolerance, we combine results for those events. The errors introduced by these approximations, as well as storage requirements are currently under investigation.

B. Half-State Models

Based on very preliminary results, it appears that the recursive half-state model is computationally more attractive than the skew half-state model. Coding the skew model is difficult because we must expand ZGZH and $ZGZ\underline{g}$ in Eq.(66). We do not know general formulas for these complicated operator matrices. The recursive model, on the other hand, can be coded directly from Eqs.(45) and (46), making use of the very simple nature of $TZ^{-2}T\underline{x}(t)$, as given by Eq.(50). Either of the two approaches, described above in Paragraph A can be used to simulate the recursive model. We are presently studying such approaches.

VII. CONCLUSIONS

We have developed state space models for lossless layered media which are described by the wave equation and boundary conditions. Our models are for non-equal one-way travel times, and are therefore more general than traditional transfer function models, which are usually for layers of equal one-way travel times. Our state space models represent a new class of equations, which we call causal functional equations. These equations are linear, time-invariant, continuous-time equations with multiple time delays. The impulse response of our system is a sequence of unequally spaced impulse functions.

We have developed full-state models, which require 2K states for a K-layer media system, and, have also developed half-state models, which require only K states. Additionally, we have generalized certain transfer functions, which appear in the geophysics

*This procedure has been developed by Mr. Michael Chan, a graduate student in the Electrical Engineering Department at the University of Southern California, and is reported on in more detail in Ref. 17.

literature, from layered media with equal travel times to layered media with unequal travel times.

Now that we have developed state space models for a K-layer media system, much work remains before us. Since we have been led to a new class of equations, causal functional equations, they must be studied not only from a simulation point of view, but also from a system theoretic point of view. Efficient computational methods must be developed, and notions such as observability, controllability, and identifiability must be expanded to this new class of equations. Work is presently underway in these areas.

Additional areas of study, which are also underway, all deal with what can be done with the state space models. These studies include: (1) extending the model to include absorption and non-normal incidence effects; (2) identifying reflection coefficients and travel times using a recursive layer-stripping procedure; and (3) developing minimum-variance state estimators.

In closing, we wish to summarize a very interesting decomposition of output, y(t), which was developed by Mendel (Ref.15), but is originally due to Bremmer (Ref.18). This decomposition was very easy to develop using our state space models. The decomposition, which we refer to as a canonical Bremmer Series decomposition, is: the complete output, y(t), from a K-layer media system, which is comprised of the superposition of primaries, secondaries,* tertiaries, etc., can be obtained from a single state space model of order 2K — the complete model — or from an infinite number of models, each of order 2K, the output of the first of which is just the primaries, the output of the second of which is just the secondaries, etc.

By thinking of y(t) as the superposition of its constituents — primaries, secondaries, etc., — we can write it as

$$y(t) = \sum_{j=1}^{\infty} y_j(t) \; . \tag{78}$$

Equation (78) is the Bremmer Series decomposition of y(t). Bremmer shows how to compute the constituents, $y_j(t)$, from integral equations which relate $y_j(t)$ to $y_{j-1}(t)$. Mendel, on the other hand, shows how to compute the $y_j(t)$ as depicted in Fig.7. Input m(t) drives a state space primaries model, whose upgoing states drive a state space secondaries model, etc. The D-U full-state model is most appropriate for characterizing the Bremmer Series decomposition. Further details on the structure of the n-aries model as well as a proof of validity of the decomposition are given in Ref.15.

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*n-aries result from exactly 2n-1 reflections from interfaces within a layered media system.

APPENDIX A. Derivation of Layer Transfer Functions

We direct our attention at the upgoing and downgoing states in each layer [Eq.(9)]. It is straightforward to show that the complete set of 2K Laplace transformed state equations can be written, as

$$\left.\begin{aligned} F_1\underline{X}_1 &= G_1M+H_1\underline{X}_2 \\ F_2\underline{X}_2 &= G_2\underline{X}_1+H_2\underline{X}_3 \\ &\;\;\cdots \\ F_{K-1}\underline{X}_{K-1} &= G_{K-1}\underline{X}_{K-2}+H_{K-1}\underline{X}_K \\ F_K\underline{X}_K &= G_K\underline{X}_{K-1} \end{aligned}\right\} \tag{A-1}$$

where explicit dependence of all quantities on s has been omitted for notational simplicity. Matrices F_i, H_i, and G_i are defined in Eqs.(24), (25), and (26) and (27), respectively. From the last equation in (A-1), we see that

$$\underline{X}_K = F_K^{-1}G_K\underline{X}_{K-1} \triangleq W_K^{-1}G_K\underline{X}_{K-1} \; . \tag{A-2}$$

Substitute this equation into the next to the last equation in (A-1), to show that

$$\underline{X}_{K-1} = (F_{K-1}-H_{K-1}W_K^{-1}G_K)^{-1}G_{K-1}\underline{X}_{K-2} \tag{A-3}$$

which can also be written, as

$$\underline{X}_{K-1} = W_{K-1}^{-1}G_{K-1}\underline{X}_{K-2} \tag{A-4}$$

where

$$W_{K-1} \triangleq F_{K-1}-H_{K-1}W_K^{-1}G_K \; . \tag{A-5}$$

Proceeding in a similar manner, it is easily shown that Eq.(A-1) can be expressed in terms of matrix W_i, as

$$\left.\begin{aligned} \underline{X}_1 &= W_1^{-1}G_1M \\ \underline{X}_2 &= W_2^{-1}G_2\underline{X}_1 \\ &\;\;\cdots \\ \underline{X}_K &= W_K^{-1}G_K\underline{X}_{K-1} \end{aligned}\right\} \tag{A-6}$$

where W_i is defined in Eq.(23).

In order to obtain the state transfer function in Eq.(21), substitute $\underline{X}_1$ into $\underline{X}_2$, $\underline{X}_2$ into $\underline{X}_3$, ..., and $\underline{X}_{K-1}$ into $\underline{X}_K$. In order to obtain the reflection transfer function Eq.(22), first express Y(s), from Eq.(3), in terms of $\underline{X}_1(s)$, as

$$Y(s) = (1-r_0)(1,0)\underline{X}_1(s)+r_0M(s) \; , \tag{A-7}$$

and then substitute $\underline{X}_1$, from Eq.(A-6), into Eq.(A-7).

APPENDIX B. Derivation of Recursive Half-State Model

For the purposes of deriving the recursive half-state model, it is convenient to use a recursive transfer function comparable to Eq.(38) which iterates in a forward direction instead of a backwards direction. In this way, a one-layer system will be associated with subscripted "one" quantities rather than subscripted "K" quantities, etc. Consider Eq. (38) for j=0:

$$R_0(s) = \frac{r_0+z_1^2R_1(s)}{1+r_0z_1^2R_1(s)} \; . \qquad \text{(B-1)}$$

Renumber the layers of our K-layer media system as depicted in Fig.8b, in which the top layer is now the Kth layer. Making the transformations $r_0 \to r_K$, $z_1 \to z_K$, $R_0(s) \to \overline{R}_K(s)$, and $R_1(s) \to \overline{R}_{K-1}(s)$, Eq.(B-1) can be written as

$$\overline{R}_K(s) = \frac{r_K+z_K^2\overline{R}_{K-1}(s)}{1+r_Kz_K^2\overline{R}_{K-1}(s)} \; . \qquad \text{(B-2)}$$

Reflection transfer function $\overline{R}_K(s)$ can be obtained in a recursive manner from

$$\left.\begin{aligned} \overline{R}_j(s) &= \frac{r_j+z_j^2\overline{R}_{j-1}(s)}{1+r_jz_j^2\overline{R}_{j-1}(s)} \; , \; j=1,2,\ldots,K \\ \overline{R}_0(s) &= r_0 \; . \end{aligned}\right\} \qquad \text{(B-3)}$$

We prove Theorem 1 by first stating and proving the truth of a recursive half-state model for the layer-reversed system in Fig.8b, and then transforming variables from that model to the Fig.8a situation.

Theorem 3. For the K-layer media system depicted in Fig.8b, let

$$\underline{x}^j(t) = \mathrm{col}(x_1(t),x_2(t),\ldots,x_j(t)) \qquad \text{(B-4)}$$

and

$$Z_j = \mathrm{diag}(z_1,z_2,\ldots,z_j) \qquad \text{(B-5)}$$

where $j=1,2,\ldots,K$. Then,

$$Z_j^{-2}\underline{x}^j(t) = A^j\underline{x}^j(t)+\underline{b}^jm_j(t) \qquad \text{(B-6)}$$

$$y^j(t) = \underline{c}'^j\underline{x}^j(t)+r_jm_j(t) \qquad \text{(B-7)}$$

where $m_j(t)$ and $y^j(t)$ denote the input and output, respectively, at the surface of a j-layer media system, and

$$\left.\begin{aligned} A^1 &= -r_0r_1 \\ b^1 &= r_0 \\ c^1 &= (1-r_1^2) \end{aligned}\right\} \qquad \text{(B-8)}$$

$$A^j = \left(\begin{array}{c|c} A^{j-1} & -r_j\underline{b}^{j-1} \\ \hline \underline{c}'^{j-1} & -r_jr_{j-1} \end{array}\right) \qquad \text{(B-9)}$$

$$\underline{b}^j = \left(\begin{array}{c} \underline{b}^{j-1} \\ \hline r_{j-1} \end{array}\right) \qquad \text{(B-10)}$$

and

$$\underline{c}^j = \mathrm{col}(\underbrace{0,0,\ldots,0}_{j-1},(1-r_j^2)) \; . \qquad \text{(B-11)}$$

Proof: Our proof, which is by induction, is based on the block diagram decomposition of Eq.(B-3), depicted in Fig.9. To begin, we develop the initialization in Eq.(B-8); then, we demonstrate the truth of the theorem for j=2; and, finally, assuming the truth of the theorem for j-1, we demonstrate its truth for j. Because the steps for j=2 and j are so similar in nature, we omit the j=2 proof.

(1) Initializations (j=1)

From Fig.9, we find, for j=1, that $[\overline{R}_0(s) = r_0]$

$$z_1^{-2}x^1(t) = -r_0r_1x^1(t)+r_0m_1(t) \qquad \text{(B-12)}$$

and

$$y^1(t) = (1-r_1^2)x^1(t)+r_1m_1(t) \; . \qquad \text{(B-13)}$$

Comparing Eqs.(B-12) and (B-13) with Eqs.(B-6) and (B-7), respectively, it is clear that A^1, b^1, and c^1 are as defined in Eq.(B-8).

(2) Inductive Step

Observe that the model of the (j-1)-layer media system is embedded in the forward path of the feedback loop in the model for the j-layer media system. We have labeled the input and output points of the $\overline{R}_{j-1}(s)$ block as $M_{j-1}(s)$ and $Y^{j-1}(s)$, respectively. Observe, that

$$z_j^{-2}x^j(t) = y^{j-1}(t) \; ; \qquad \text{(B-14)}$$

but, by assumption, $y^{j-1}(t)$ satisfies Eq.(B-7); hence

$$z_j^{-2}x^j(t) = \underline{c}'^{j-1}\underline{x}^{j-1}(t)+r_{j-1}m_{j-1}(t) \; . \qquad \text{(B-15)}$$

From Fig.9, however

$$m_{j-1}(t) = m_j(t)-r_jx^j(t) \qquad \text{(B-16)}$$

whereupon Eq.(B-15) becomes

$$z_j^{-2}x^j(t) = \underline{c}'^{j-1}\underline{x}^{j-1}(t)-r_jr_{j-1}x^j(t)+r_{j-1}m_j(t) \qquad \text{(B-17)}$$

which is our state equation for x^j.

Next, we must combine Eq.(B-17) with our assumed state equation for $\underline{x}^{j-1}$, Eq.(B-6); but, we must replace $m_{j-1}(t)$ in the latter equation by Eq.(B-16). The equation for $\underline{x}^{j-1}$ is

$$Z_{j-1}^{-2}\underline{x}^{j-1}(t) = A^{j-1}\underline{x}^{j-1}(t)-r_j\underline{b}^{j-1}x^j(t)+\underline{b}^{j-1}m_j(t) \; . \qquad \text{(B-18)}$$

Clearly now, Eqs.(B-18) and (B-17) can be combined and expressed as in Eq.(B-6), where A^j and $\underline{b}^j$ are given by Eqs.(B-9) and (B-10), respectively.

Finally, from Fig.6, we see that

$$y^j(t) = (1-r_j^2)x^j(t)+r_jm_j(t) \qquad \text{(B-19)}$$

and, this can be expressed as in Eq.(B-7), where $\underline{c}^j$ is defined in Eq.(B-11). ■

While Eqs.(B-9) and (B-10) are interesting in their own right, they do not reveal the intrinsic detailed structures of A^j and $\underline{b}^j$. It is a straightforward matter to iterate these two equations, using their starting values from Eq.(B-8), to show that

$$\underline{b}^j = \text{col}(r_0, r_1, \ldots, r_{j-1}) \tag{B-20}$$

and

$$A^j =$$

$$\begin{pmatrix} -r_0r_1 & -r_0r_2 & -r_0r_3 & \cdots & -r_0r_{j-1} & -r_0r_j \\ (1-r_1^2) & -r_1r_2 & -r_1r_3 & \cdots & -r_1r_{j-1} & -r_1r_j \\ 0 & (1-r_2^2) & -r_2r_3 & \cdots & -r_2r_{j-1} & -r_2r_j \\ 0 & 0 & (1-r_3^2) & \cdots & -r_3r_{j-1} & -r_3r_j \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (1-r_{j-1}^2) & -r_{j-1}r_j \end{pmatrix}. \tag{B-21}$$

<u>Proof of Theorem 1</u>: From Fig.8, we see that the state space model for the desired system, Fig.8a, can be obtained from the state space model for the layer-reversed system, Fig.8b, by means of the following transformation of variables:

$$z_j \to z_{K-j+1} \qquad j=1,2,\ldots,K \tag{B-22}$$

and

$$r_i \to r_{K-i} \qquad i=0,1,\ldots,K . \tag{B-23}$$

In order to obtain A, $\underline{b}$, and $\underline{c}$ in Eqs.(49),(47), and (48), respectively, apply Eq.(B-23) to Eqs.(B-21), (B-20), and (B-11), and, then set j=K. In order to obtain the left-hand side of Eq.(45), apply Eq.(B-22) to Z_j^{-2} [in Eq.(B-6)] and then set j=K. It is, of course understood that quantities in Theorem 3 which are superscripted or subscripted "K" [i.e., $\underline{x}^K(t)$] are the same as those unsuperscripted or unsubscripted quantities in Theorem 1.

APPENDIX C. Derivation of Relationships between $\underline{x}_{s1}(t)$ and $\underline{u}(t)$

Here, we shall show that there exist permutation matrices L_a and L_b, whose dimensions and structures depend on whether K is even or odd, such that

$$\underline{u}(t) = L_a'L_a\underline{x}_{s1}(t) + L_b'L_b\underline{x}_{s2}(t) . \tag{C-1}$$

Substitute Eq.(62b), for $\underline{x}_{s2}(t)$, into Eq.(C-1) to show that

$$\underline{u}(t) = (L_a'L_a + L_b'L_bZH)\underline{x}_{s1}(t) + L_b'L_bZ\underline{g}m(t) . \tag{C-2}$$

It is straightforward to show, by direct substitution of the defining equations for L_b, Z, and $\underline{g}$, that $L_b'L_bZ\underline{g} = 0$; hence,

$$\underline{u}(t) = (L_a'L_a + L_b'L_bZH)\underline{x}_{s1}(t) \tag{C-3}$$

which is the desired result.

Now we return to the decomposition of $\underline{u}(t)$ in Eq.(C-1). Let $\underline{x}_{s1}^*(t)$ and $\underline{x}_{s2}^*(t)$ denote rearranged $\underline{x}_{s1}(t)$ and $\underline{x}_{s2}(t)$ vectors. For <u>K even</u>,

$$\underline{x}_{s1}^*(t) \triangleq \text{col}(u_1(t), u_3(t), \ldots, u_{K-1}(t) | d_2(t), \ldots, d_K(t)) = \text{col}(\underline{u}_{s1}^*(t) | \underline{d}_{s1}^*(t)) \tag{C-4}$$

and

$$\underline{x}_{s2}^*(t) \triangleq \text{col}(u_2(t), u_4(t), \ldots, u_K(t) | d_1(t), \ldots, d_{K-1}(t)) = \text{col}(\underline{u}_{s2}^*(t) | \underline{d}_{s2}^*(t)) . \tag{C-5}$$

It is straightforward to show that

$$\underline{u}_{s1}^*(t) = L_a\underline{x}_{s1}(t) \tag{C-6}$$

and

$$\underline{u}_{s2}^*(t) = L_b\underline{x}_{s2}(t) \tag{C-7}$$

where L_a and L_b are both $K/2 \times K$ matrices, and

$$L_a = (\underline{e}_1, \underline{0}, \underline{e}_2, \underline{0}, \ldots, \underline{e}_{K/2}, \underline{0}) \tag{C-8}$$

and

$$L_b = (\underline{0}, \underline{e}_1, \underline{0}, \underline{e}_2, \ldots, \underline{0}, \underline{e}_{K/2}) . \tag{C-9}$$

For <u>K odd</u>,

$$\underline{x}_{s1}^*(t) \triangleq \text{col}(u_1(t), u_3(t), \ldots, u_K(t) | d_2(t), d_4(t), \ldots, d_{K-1}(t)) = \text{col}(\underline{u}_{s1}^*(t) | \underline{d}_{s1}^*(t)) \tag{C-10}$$

and

$$\underline{x}_{s2}^*(t) \triangleq \text{col}(u_2(t), u_4(t), \ldots, u_{K-1}(t) | d_1(t), d_3(t), \ldots, d_K(t)) = \text{col}(\underline{u}_{s2}^*(t) | \underline{d}_{s2}^*(t)) . \tag{C-11}$$

Equations (C-6) and (C-7) are applicable in this case also; but, now L_a is $(\frac{K+1}{2}) \times K$ and L_b is $(\frac{K-1}{2}) \times K$, and

$$L_a = (\underline{e}_1, \underline{0}, \underline{e}_2, \underline{0}, \ldots, \underline{0}, \underline{e}_{\frac{K+1}{2}}) \tag{C-12}$$

and

$$L_b = (\underline{0}, \underline{e}_1, \underline{0}, \underline{e}_2, \ldots, \underline{e}_{\frac{K-1}{2}}, \underline{0}) . \tag{C-13}$$

In Eqs.(C-8) and (C-9), $\underline{e}_j$ are $\frac{K}{2} \times 1$ unit vectors; but, in Eq.(C-12), $\underline{e}_j$ are $\frac{K+1}{2} \times 1$, and, in Eq.(C-13), $\underline{e}_j$ are $\frac{K-1}{2} \times 1$.

Regardless of whether K is even or odd, one can show that

$$\underline{u}(t) = L_a'\underline{u}_{s1}^*(t) + L_b'\underline{u}_{s1}^*(t) . \tag{C-14}$$

Substitute Eqs.(C-6) and (C-7) into Eq.(C-14) to obtain the assumed decomposition for $\underline{u}(t)$ in Eq.(C-1).

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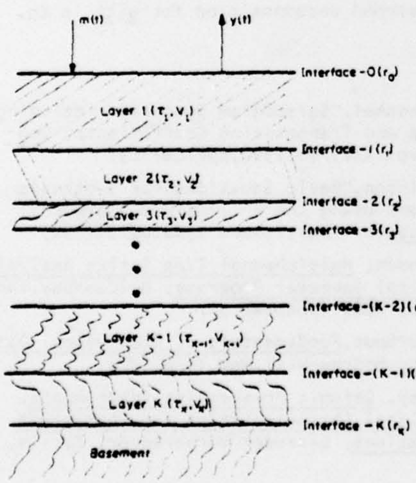

Figure 1. System of K Layered Media

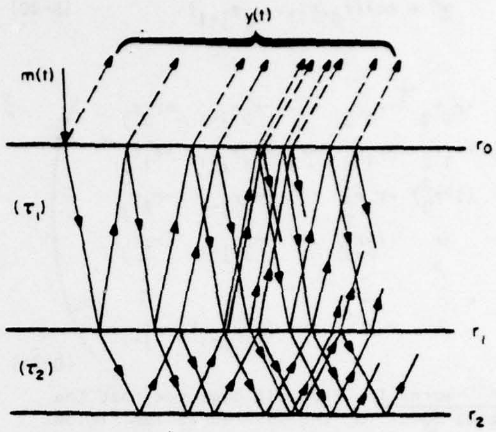

Figure 2. Two Layer Example

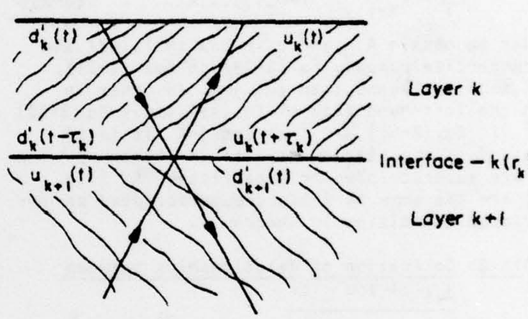

Figure 3. Reflected and Transmitted Waves at Interface k. From Eq.(8), $d_k'(t-\tau_k) \triangleq d_k(t)$.

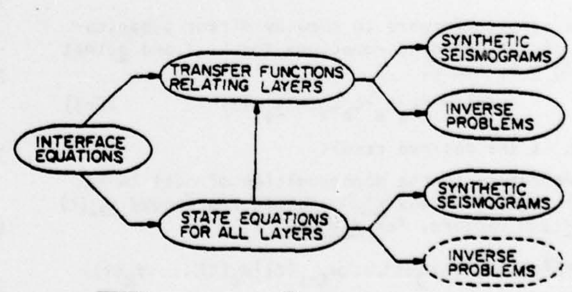

Figure 4. Summary of Past (top path) and Present (lower path) Work. Dashed Block Denotes Work too Preliminary to be Reported on in this Paper.

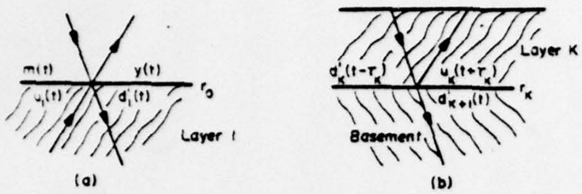

Figure 5. Reflected and Transmitted Waves at (a) Surface and (b) Interface K.

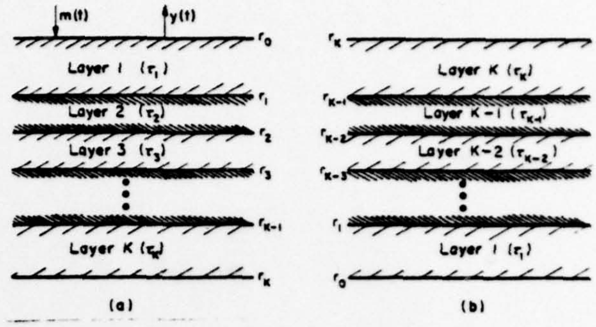

Figure 8. System of K layered media: (a) first layer is Layer 1 and the last layer is Layer K; (b) first layer is Layer K and the last layer is Layer 1.

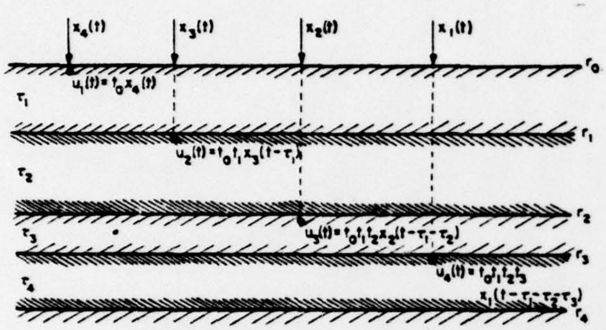

Figure 6. Relationship between $\underline{x}(t)$ and $\underline{u}(t)$. Upgoing states are located (circles) at top of each layer. Dashed lines denote transmission paths for x states. States are separated along the horizontal axis for purposes of clarity.

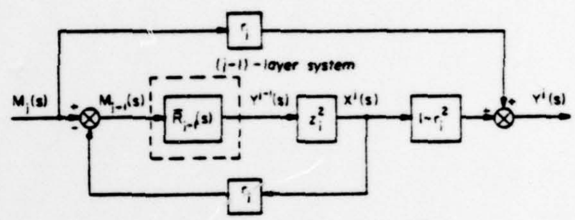

Figure 9. Block Diagram Representation of $\bar{R}_j(s)$ for a j-layer system, j=1,2, ...,K. For j=1, disregard the $M_0(s)$ and $Y^0(s)$ labels.

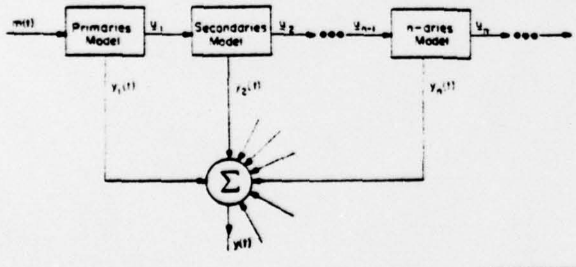

Figure 7. Canonical Bremmer Series decomposition of a seismogram signal, y(t). Vector $\underline{u}_n$ denotes the collection of K upgoing states from the n-aries model.